REDUCED DYNAMICAL MODEL OF THE VIBRATIONS OF A METAL PLATE

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ABSTRACT

The Proper Orthogonal Decomposition (POD) method is applied to the vibrations analysis of a metal plate. The data obtained from the metal plate under vibrations were measured with a laser vibrometer. The metal plate was subject to vibrations with an electrodynamical shaker in a range of frequencies from 100 to 5000 Hz. The deformation measurements were taken on a quarter of the plate in a rectangular grid of 7 x 8 points. The plate deformation measurements were used to calculate the eigenfunctions and the eigenvalues. It was found that a large fraction of the total energy of the deformation is contained within the first six POD modes. The essential features of the deformation are thus described by only the six first eigenfunctions. A reduced order model for the dynamical behavior is then constructed using Galerkin projection of the equation of motion for the vertical displacement of a plate.

1. INTRODUCTION

This paper is motivated by our attempt to provide a method for vibration deformation data analysis aimed at obtaining low–dimensional approximate description of the behavior of a metal plate under vibrations. The vibration deformation data is obtained experimentally using laser vibrometer.

Recent developments in the statistical technique of Proper Orthogonal Decomposition (POD) seems to offer some hope to capture the spatial as well as temporal behavior of the dominant features in a variety of dynamical structures. Data analysis using POD is often conducted to extract “mode shapes” or basis functions from experimental data for subsequent use in Galerkin projections that yield low-dimensional models [1]. This enables one to build efficient reduced order models based on the first few dynamically important POD modes, thus serving as potential substitute for computationally intensive simulations.

The POD method was originally suggested by Lumely [2] to extract organized large-scale structures from turbulent flows. The method provides a set of optimized orthonormal basis functions for an ensemble of data. The most important property of POD is its optimality in the sense that it provides the most efficient way of capturing the dominant features of an infinite dimensional process with only few functions.

The goal of this work is to show the feasibility of applying the Proper Orthogonal Decomposition (POD) method together with Galerkin projection to the vibration deformations. We focus on the analysis of the vibrations of a metal square plate clamped

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at its corners. This configuration was selected because of the simplicity of the problem. However, the procedure can be applied to more complicated configurations and it is the aim of further research on this subject.

2. THE EXPERIMENT.

The data being examined consists of the velocity displacement of a metal plate induced to vibration. The aluminum square plate (35×35×0.8 cm) is positioned horizontally and clamped at its four corners to an optical table, see figure 1.

A combination of an electrodynamical shaker and a power amplifier were used to induce vibration to the plate. The electrodynamical shaker was located under the plate and clamped centrally by a screw.

The symmetry of the problem allows us to examine only one quarter of the plate. Experimental data was collected at points located on a 7×8 grid with a spacing of 2 cm. This spacing enables us to resolve the spatial features of the resonant modes.

The measurement of deformation was done with a Polytec laser vibrometer connected to a controller. The laser vibrometer let us to acquire temporal history series with a temporal resolution of 10 ns and deformation resolution measurements of 1 nm. The visualization and data collection was done through a Tektronix oscilloscope. Figure 1 gives detail of the experimental set-up.

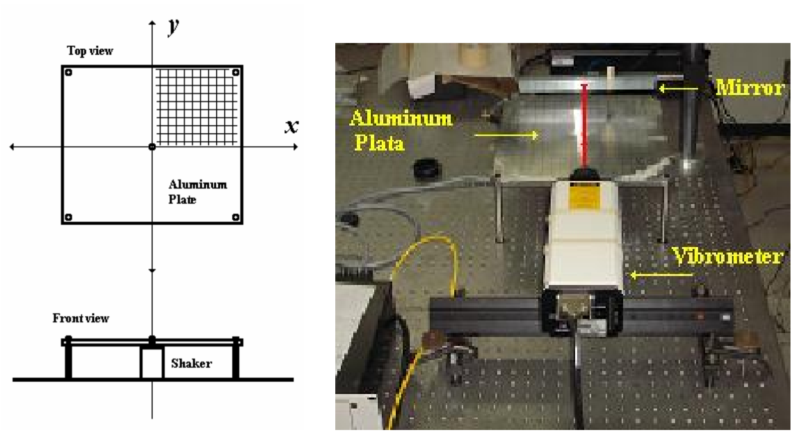

Figure 1. Set up for experimental data acquisition.

The procedure started by searching the modal frequencies of the specimen. The electrodynamical shaker was driven with a sinusoidal signal of 2 V produced by a signal generator. The frequency of the shaker was varied from 100 Hz up to 5000 Hz. The signal was sampled at 25 kHz and each data set consisted of 10,000 points. It was found that the first modal frequency was located at 175 Hz and three more resonant frequencies at 789, 2270 and 4290 Hz.

To be able to observe the modal frequency patterns some sugar was sprinkled on the plate. The characteristic vibrations give rise to lines of minimal motion (nodal regions), which separate domains containing regions of maximal vibration. The patterns arise because the sugar collects in regions of minimal vibration. In figure 2 the patterns are depicted for the four resonant modes found before.

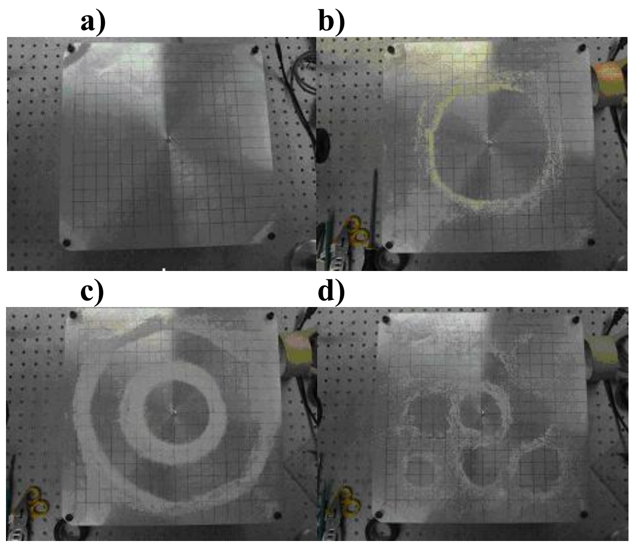

Figure 2. Resonant modes patterns at frequencies, a) 175 Hz, b) 789 Hz, c) 2270 Hz and d) 4290 Hz**.**

Once the modes were found, the plate was subject to a mix of frequencies in the range of 100 Hz to 5000 Hz. Figure 4 shows a time series and the power spectra of measurement done at an arbitrary point on the plate for this case. The figure depicts four discrete peaks corresponding to the resonant modes found previously. With these last conditions the deformation was measured at each point of the grid (7×8 points) drawn on the plate.

3. THE POD METHOD

Before proceeding, some remarks on notation are introduced. Here $\boldsymbol{x}$ and $\boldsymbol{u}$ are vectors representing a Cartesian coordinate system (x, y, z), and deformation components along Cartesian direction (u, v, w) respectively. Any time variable describing a parameter such as vibration deformation, $\boldsymbol{u}$ ($\boldsymbol{x}$, t), can be defined as a composition of a mean and a time-varying component, as given below

$$\boldsymbol{u}(\boldsymbol{x},\boldsymbol{t}) = \overline{\boldsymbol{u}} + \boldsymbol{u}'(\boldsymbol{x},\boldsymbol{t}) \quad (1)$$

where $\overline{\boldsymbol{u}}$ and $\boldsymbol{u}'$ are the mean and the varying component respectively. In this work the vertical deformation, w is considered only. Thus, the POD method is calculated only for the varying component of the vertical deformatio; $w'(x,t) = w(x,t) - \overline{w}$.

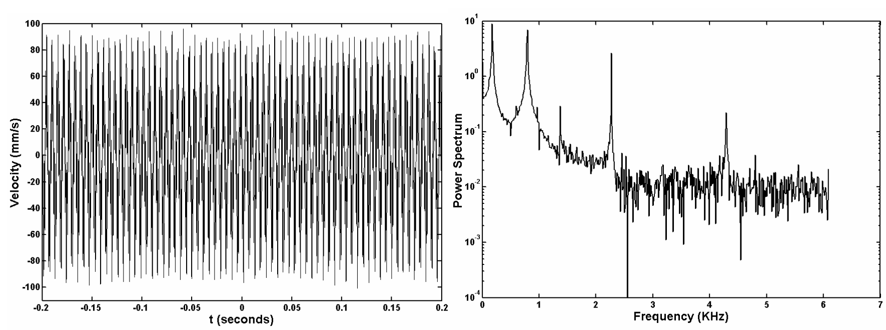

Figure 3. Time series and power spectra obtained from the plate by using a range of frequencies.

The proper orthogonal decomposition consists of finding a series of eigenfunctions ϕ (hereafter eigefunction and POD mode are used indistinctly referring to the same concept), with a respective energy value, λ, from a data set [1]. The core of the method is to solve the integral eigenvalue equation,

$$\iint_D R(x,x')\,\phi^n(x')\,dx' = \lambda^n \phi^n \tag{2}$$

where D is the two-dimensional domain where the vibration deformation takes place and the kernel, R, is the average autocorrelation function expressed as

$$R(x,x') = \langle w(x)\,w(x')\rangle \tag{3}$$

The symbol, $\langle\ \rangle$, in equation (3) represents temporal average and the superscript n in equation (2) denotes mode number.

The solution of equation (1) represents a set ϕ^n of basis functions with special properties attractive for the purpose of deriving dynamical equations via Galerkin projection. The eigenfunctions form an orthogonal system, which can be normalized as

$$(\phi^m, \phi^n) = \delta_{mn}\lambda_n \tag{4}$$

where δ_{mn} is the Kronecker delta symbol. The basis functions, ϕ^n, is complete in the sense that, w**,** is represented as an expansion of orthogonal eigenfuctions

$$w(x,t_k) = \sum_{n=1}^{N} \varsigma^n(t_k)\phi^n(x) \tag{5}$$

where ς is the dot product for w and ϕ^n. That is, ς is the projection of w in the direction represented by ϕ^n. An important property of eigenfuction ϕ^n is that it can be expanded as a linear combination of the instantaneous values of w as

$$\phi(x) = \sum_{k=1}^{M-1} A_k w(x,t_k) \tag{6}$$

where, the eigenfunction ϕ possess all the properties of the vertical deformation vector, w.

In order to avoid a time consuming procedure by using the direct method of POD, the snapshot POD proposed by Sirovich [1] has been used to calculate eigenfunctions and eigenvalues. Then, R can be expressed as

$$R(x,x') = \frac{1}{M}\sum_{j=1}^{M} w^j(x)w^j(x') \tag{7}$$

Here, M is the number of snapshots (or realizations).

If equations (2), (6) and (7) are combined, the following is obtained

$$CA = \lambda A \tag{8}$$

where C is an $M \times M$ matrix defined as

$$C_{ij} = (w_i, w_j) \tag{9}$$

4. THE VERTICAL DISPLACEMENT EQUATION

We are now interested in deriving dynamical equations for the evolution of the expansion coefficients $\varsigma(t)$ in time. A common process is the method of Galerkin projection [1]. The Galerkin method is a discretization scheme for Partial Differential

Equations (PDE's), which is generically categorized as one of the spectral methods or methods of weighted residuals. This method is based on the separation of variables approach and is an attempt to find an approximate solution in the form of truncated series expansion given by

$$w(x,t_k)=\sum_{n=1}^{N}\varsigma^n(t_k)\phi^n(x) \tag{10}$$

Here, ϕ^n are known eigenfunctions calculated using the POD method described earlier. Thus, the original infinite-dimensional system is approximated by a N-dimensional system, where the order of the reduced model is determined by the truncation of the vertical deformation, w. The method then involves the projection of the truncated vertical deformation represented by equation (10) on to the vertical displacement equation.

In this work a square plate with side length l will be considered. The displacement of the plate in, z direction is denoted by $w = w(x, y, t)$, in which t is time. The following symbols will be used: μ is the mass of the plate per unit area perpendicular to z-axis, ρ is the mass density of the plate, A is the area of the cross section of the plate perpendicular to the x-axis, (so $A = lh$, where h is the thickness of the plate), E is the elasticity modulus, I is the moment of inertia of the cross section with respect to the x axis. We neglect internal damping and consider the weight W of the plate per unit area to be constant ($W = \mu g$, g is the gravitational acceleration). No other external forces are assumed to be present. The equation of motion for the vertical displacement of the plate is given by

$$\frac{\partial^2 w}{\partial t^2}+\frac{EI}{\mu}\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+\frac{k}{\mu}w+\frac{b}{\mu}w^2=-g \tag{11}$$

Where k and b are spring constants. Using dimensionless variables to facilitate de analysis:

$$\tilde{w}=\frac{l}{A}w,\ \tilde{x}=\frac{x}{l},\ \tilde{y}=\frac{y}{l},\ \tilde{t}=\frac{1}{l^2}\left(\frac{EI}{\mu}\right)^{1/2}t$$

thus, the equation (11) becomes

$$\frac{\partial^2\tilde{w}}{\partial\tilde{t}^2}+\frac{\partial^4\tilde{w}}{\partial\tilde{x}^4}+\frac{\partial^4\tilde{w}}{\partial\tilde{x}^2\partial\tilde{y}^2}+\frac{\partial^4\tilde{w}}{\partial\tilde{y}^4}+\frac{l}{EI}\left(k\tilde{w}+\frac{bA}{l}\tilde{w}^2+\frac{lw}{A}g\right)=0 \tag{12}$$

By projecting the dynamical equation (12) of the vertical displacement onto the function basis defined by the eigenfunction ϕ, using Galerkin procedure a system of ordinary differential equations for the evolution in time of the expansion coefficients ζ can be obtained, which is of the form:

$$\frac{\partial^2\varsigma_k}{\partial\tilde{t}^2}=A_k+\sum_{i=1}^{N}\left(B_{ik}\varsigma_i+C_{ik}\varsigma_i^2\right) \tag{13}$$

where the constants A, B and C are defined as:

$$A_k=-\left\langle\frac{\partial^4\overline{\tilde{w}}}{\partial\tilde{x}^4}+\frac{\partial^4\overline{\tilde{w}}}{\partial\tilde{x}^2\partial\tilde{y}^2}+\frac{\partial^4\overline{\tilde{w}}}{\partial\tilde{y}^4}+\frac{l^4}{EI}\left(k\overline{\tilde{w}}+\frac{bA}{l}\overline{\tilde{w}}^2+\frac{l\mu}{A}g\right),\phi^k\right\rangle$$

$$B_{ik}=-\left\langle\frac{\partial^4\phi^i}{\partial\tilde{x}^4}+\frac{\partial^4\phi^i}{\partial\tilde{x}^2\partial\tilde{y}^2}+\frac{\partial^4\phi^i}{\partial\tilde{y}^4}+\frac{l^4}{EI}\left(k\phi^i+\frac{2bA}{l}\overline{\tilde{w}}\phi^i\right),\phi^k\right\rangle$$

$$C_{ik} = -\left\langle \frac{l^4}{EI}\left(\frac{bA}{l}\left(\phi^i\right)^2\right), \phi^k \right\rangle$$

The constants *A*, *B* and *C* are determined by using calculated eigenfuctions as explained above.

5. RESULTS AND DISCUSSIONS

It is important to mention that the laser vibrometer measurements taken from the plate are not in phase. Thus, it was taken special care to assure that the beginning of each time series was a maximum. We select from each time series 600 points of the 10000 measurements. It was observed that it was not needed to add more points to the data set. For each time measurement a snapshot was constructed, and a total of 600 snapshots were constructed. To have a full view of the entire plate we projected the results obtained in a quarter of the plate to the other three quadrants. In order to enhance the results, each snapshot is interpolated on a grid of 61X71 by using linear polynomial interpolation provided by MatLab.

Equation (8) is used to calculate eigenfunctions and eigenvalues. The 600 snapshots were used to apply the POD method. Therefore, 600 POD modes were obtained with their respective eigenvalues.

Table I shows the relative energy value in percentage for the first 6 POD modes. It is clear that most of the energy is contained in the first eigenvalue of the POD modes. Very little energy is contained at higher POD modes. This eigenvalue is very energetic because it is related to the first resonant mode. We have a 99.9 percent of the total energy contained in the first six POD modes.

Table 1. Relative energy for the first six POD modes

Eigenvalue(λ^n)	Energy$\left(\lambda^n \Big/ \sum_{n=1}^{M} \lambda^n\right)$
1	98.0051
2	1.7112
3	0.2103
4	0.0366
5	0.0116
6	0.0085

In the Figure 4 the first four POD modes shape are plotted. The first and third POD modes correspond to the first and second resonant modes found previously. The second and four POD modes have shape unknown to the authors. However, these POD modes contribute to reconstruction of the vertical deformation, *w*.

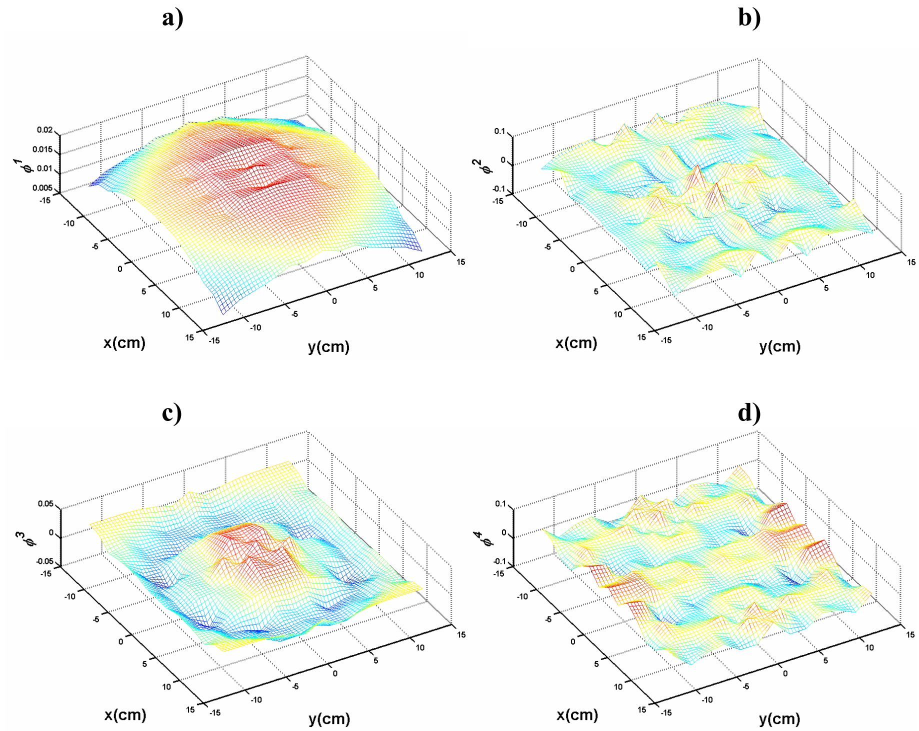

Figure 4. The first four POD modes shape.

The contributions of each eigenfunction to the original deformation, *w,* can be done by equation (5). Figure 5 shows the reconstructed and original deformation *w* of an arbitrary point on the plate. Also shown in the figure is the difference between the original and reconstructed deformation. By using the first POD mode in the reconstruction, it is clear from the figure that most of the important details are captured. Adding the first four modes in the reconstruction reproduces all the features of the original deformation, *w*.

These are preliminary results concerning the combination of the POD method and Galerkin projection. In this first work the procedure to obtain reduced order model for the behavior of a square plate subject to vibrations has been introduced. Future work will include results of the dynamical model.

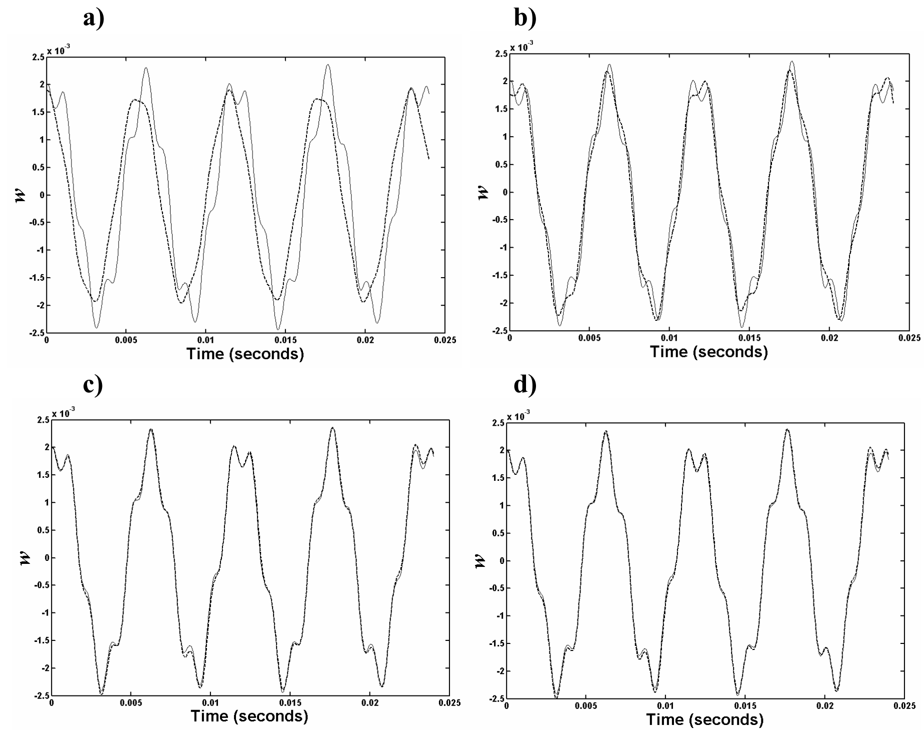

Figure 5. Comparison of the original and reconstructed deformation using different POD modes. Continuous and dashed represent original and reconstructed signal deformation respectively. Reconstructions using a) one POD mode, b) two POD modes, c) three POD modes and d) four POD modes.

6. CONCLUSIONS

The use of the POD method in the analysis of the modal analysis of an aluminum square plate was presented. The signals from the deformation taken with a laser vibrometer are used to determine the eigenfunctions and eigenvalues. The POD method shows that a single dominant eigenfunction contains most of the energy. Thus, in order to represent a signal of an arbitrary point on the plate only a few eigenfunctions are needed. Hence, we believe that the POD analysis of the vibration data is useful in determining a reduced order model to be applied to vibration control schemes with a better insight than that provided by traditional analysis methods.

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